

Inequalities among Related Triplets of Fibonacci Numbers

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Abstract

In this paper we consider Fibonacci inequalities and relate them through the sequence $\{m_r\}_{r=1}^n$ defined by where n is a fixed natural number and F_1, F_2, F_3, \dots are the ordinary Fibonacci numbers.

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INTRODUCTION

“Fibonacci inequalities” have been studied in a variety of contexts. Atanassov [3] considered the Fibonacci inequalities and relate them through the sequence $\{m_r\}_{r=0}^n$ defined by

$m_r = F_r F_{n+1-r}$, where n is a fixed natural number and F_1, F_2, F_3, \dots are the ordinary Fibonacci numbers as defined in [4]. We consider the similar Fibonacci inequalities and relate them through the sequence $\{m_r\}_{r=1}^n$ defined by $m_r = F_r F_{n+1-r} F_{n+2-r}$, where n is a fixed natural number and F_1, F_2, F_3, \dots are the ordinary Fibonacci numbers.

Theorem: For every natural number k , the following inequalities for the elements of the sequence $\{m_k\}_{k=1}^n$ are valid :

(a) For $n = 4k$,

$$(i) F_1 F_{4k} F_{4k+1} > F_3 F_{4k-2} F_{4k-1} > \dots > F_{2k+1} F_{2k} F_{2k+1} > F_{2k+2} F_{2k-1} F_{2k} > F_{2k+4} F_{2k-3} F_{2k-2} > \dots > F_{4k} F_1 F_2$$

(ii)

$$F_2 F_{4k-1} F_{4k} > F_4 F_{4k-3} F_{4k-2} > \dots > F_{2k} F_{2k+1} F_{2k+2} > F_{2k+1} F_{2k} F_{2k+1} > F_{2k+3} F_{2k-2} F_{2k-1} > \dots > F_{4k-1} F_2 F_3$$

(b) For $n = 4k+1$,

(i)

$$F_1 F_{4k+1} F_{4k+2} > F_3 F_{4k-1} F_{4k} > \dots > F_{2k+1} F_{2k+1} F_{2k+2} > F_{2k+2} F_{2k} F_{2k+1} > F_{2k+4} F_{2k-2} F_{2k-1} > \dots > F_{4k} F_2 F_3$$

(ii)

$$F_2 F_{4k} F_{4k+1} > F_4 F_{4k-2} F_{4k-1} > \dots > F_{2k} F_{2k+2} F_{2k+3} > F_{2k+1} F_{2k+1} F_{2k+2} > F_{2k+3} F_{2k-1} F_{2k} > \dots > F_{4k+1} F_1 F_2$$

(c) For $n = 4k+2$,

(i)

$$F_1 F_{4k+2} F_{4k+3} > F_3 F_{4k} F_{4k+1} > \dots > F_{2k+1} F_{2k+2} F_{2k+3} > F_{2k+2} F_{2k+1} F_{2k+2} > F_{2k+4} F_{2k-1} F_{2k} > \dots > F_{4k+2} F_1 F_2$$

(ii)

$$F_2 F_{4k+1} F_{4k+2} > F_4 F_{4k-1} F_{4k} > \dots > F_{2k} F_{2k+3} F_{2k+4} > F_{2k+1} F_{2k+2} F_{2k+3} > F_{2k+3} F_{2k} F_{2k+1} > \dots > F_{4k+1} F_2 F_3$$

(d) For $n = 4k+3$,

(i)

$$F_1 F_{4k+3} F_{4k+4} > F_3 F_{4k-1} F_{4k+2} > \dots > F_{2k+1} F_{2k+3} F_{2k+4} > F_{2k+2} F_{2k+2} F_{2k+3} > F_{2k+4} F_{2k} F_{2k+1} > \dots > F_{4k+2} F_2 F_3$$

(ii)

$$F_2 F_{4k+2} F_{4k+3} > F_4 F_{4k} F_{4k+1} > \dots > F_{2k} F_{2k+4} F_{2k+5} > F_{2k+1} F_{2k+3} F_{2k+4} > F_{2k+3} F_{2k+1} F_{2k+2} > \dots > F_{4k+3} F_1 F_2$$

Proof : case (a)

$$(i) F_1 F_{4k+4} F_{4k+5} - F_3 F_{4k+2} F_{4k+3}$$

$$= F_{4k+4} F_{4k+3} + F_{4k+4}^2 - 2F_{4k+2} F_{4k+3}$$

$$\begin{aligned}
 &= F_{4k+3}^2 + F_{4k+4}^2 - F_{4k+2}F_{4k+3} \\
 &= F_{4k+4}^2 + F_{4k+3}(F_{4k+3} - F_{4k+2}) > 0
 \end{aligned}$$

Thus, $F_1F_{4k+4}F_{4k+5} > F_3F_{4k+2}F_{4k+3}$.

This shows that the inequality (1.1) is valid for $i = 1$.

$$F_{2i-1}F_{4k-2i+6}F_{4k-2i+7} > F_{2i+1}F_{4k-2i+4}F_{4k-2i+5} \quad \dots\dots\dots(1.1)$$

Let us assume that, for some i , $1 \leq i \leq k$, the inequality (1.1) is true.

For desired result, we use induction method and prove that the inequality

$$F_{2i+1}F_{4k-2i+4}F_{4k-2i+5} > F_{2i+3}F_{4k-2i+2}F_{4k-2i+3} \quad \dots\dots\dots(1.2)$$

is also true.

$$\begin{aligned}
 \text{But, } &F_{2i+1}F_{4k-2i+4}F_{4k-2i+5} - F_{2i+3}F_{4k-2i+2}F_{4k-2i+3} \\
 &= 2F_{4k-2i+3}^2F_{2i+1} + 3F_{4k-2i+3}F_{4k-2i+2}F_{2i+1} + F_{4k-2i+2}^2F_{2i+1} - F_{2i+2}F_{4k-2i+2}F_{4k-2i+3} - F_{2i+1}F_{4k-2i+2}F_{4k-2i+3} \\
 &= 2F_{4k-2i+3}^2F_{2i+1} + F_{4k-2i+2}^2F_{2i+1} + (2F_{2i+1} - F_{2i+2})F_{4k-2i+2}F_{4k-2i+3} \\
 &= 2F_{4k-2i+3}^2F_{2i+1} + F_{4k-2i+2}^2F_{2i+1} + (F_{2i+1} - F_{2i})F_{4k-2i+2}F_{4k-2i+3} > 0
 \end{aligned}$$

Now, (ii) $F_2F_{4k+3}F_{4k+4} - F_4F_{4k+1}F_{4k+2}$

$$\begin{aligned}
 &= (F_{4k+1} + F_{4k+2})(F_{4k+1} + 2F_{4k+2}) - 3F_{4k+1}F_{4k+2} \\
 &= F_{4k+1}^2 + 2F_{4k+2}^2 > 0
 \end{aligned}$$

Thus, $F_2F_{4k+3}F_{4k+4} > F_4F_{4k+1}F_{4k+2}$

This shows that the inequality (1.3) is valid for $i = 1$.

$$F_{2i}F_{4k-2i+5}F_{4k-2i+6} > F_{2i+2}F_{4k-2i+3}F_{4k-2i+4} \quad \dots\dots\dots(1.3)$$

Let us assume that for some i , $1 \leq i \leq k$ the inequality (1.3) is true.

Then we must prove that the inequality (1.4) is also true.

$$F_{2i+2}F_{4k-2i+3}F_{4k-2i+4} > F_{2i+4}F_{4k-2i+1}F_{4k-2i+2} \quad \dots\dots\dots(1.4)$$

But, $F_{2i+2}F_{4k-2i+3}F_{4k-2i+4} - F_{2i+4}F_{4k-2i+1}F_{4k-2i+2}$

$$\begin{aligned}
 &= F_{2i+2}(F_{4k-2i+1} + F_{4k-2i+2})(2F_{4k-2i+2} + F_{4k-2i+1}) - (F_{2i+1} + 2F_{2i+2})F_{4k-2i+1}F_{4k-2i+2} \\
 &= F_{2i+2}(3F_{4k-2i+1}F_{4k-2i+2} + F_{4k-2i+1}^2 + 2F_{4k-2i+2}^2) - (F_{2i+1} + 2F_{2i+2})F_{4k-2i+1}F_{4k-2i+2} > 0
 \end{aligned}$$

This shows that the inequality (1.4) is true and hence (ii) is true.

Particularly, when $k = 3$, n becomes 12 which gives the following two results:

$$1) F_1F_{12}F_{13} > F_3F_{10}F_{11} > F_5F_8F_9 > F_7F_6F_7 > F_8F_5F_6 > F_{10}F_3F_4 > F_{12}F_1F_2$$

$$2) F_2 F_{11} F_{12} > F_4 F_9 F_{10} > F_6 F_7 F_8 > F_7 F_6 F_7 > F_9 F_4 F_5 > F_{11} F_2 F_3$$

Similarly other cases can be proved.

Corollary : For every natural number n the maximal element [1] and [2] of the sequence

$\{m_k\}_{k=1}^n$ is $F_1 F_n F_{n+1}$ and the minimal element is $F_n F_1 F_2$.

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